

ON THE EXISTENCE OF A GLOBAL NEIGHBOURHOOD

TOM COATES AND HIROSHI IRITANI

ABSTRACT. Suppose that a complex manifold M is locally embedded into a higher-dimensional neighbourhood as a submanifold. We show that, if the local neighbourhood germs are compatible in a suitable sense, then they glue together to give a global neighbourhood of M . As an application, we prove a global version of Hertling–Manin’s unfolding theorem for germs of TEP structures; this has applications in the study of quantum cohomology.

1. INTRODUCTION

We prove:

Theorem 1. *Let M be a complex manifold of dimension m and let \mathcal{A} be a sheaf of \mathbb{C} -algebras over M . Suppose that there exist a natural number n and a morphism of sheaves of \mathbb{C} -algebras $\pi: \mathcal{A} \rightarrow \mathcal{O}_M$ such that for each $x \in M$ there exists an open neighbourhood U of x and an isomorphism $\mathcal{A}|_U \cong \iota^{-1}\mathcal{O}_{U \times \mathbb{C}^n}$, where $\iota: U \hookrightarrow U \times \mathbb{C}^n$ is the embedding $x \mapsto (x, 0)$, such that the following diagram commutes:*

$$(1) \quad \begin{array}{ccc} \mathcal{A}|_U & \xrightarrow{\sim} & \iota^{-1}\mathcal{O}_{U \times \mathbb{C}^n} \\ \pi|_U \searrow & & \swarrow \iota^\# \\ & \mathcal{O}_U & \end{array}$$

Here $\iota^\#$ is the canonical morphism induced by ι . Then:

- (i) *There exist a complex manifold M' of dimension $m+n$ and a closed embedding $\iota: M \rightarrow M'$ such that we have an isomorphism $\mathcal{A} \cong \iota^{-1}\mathcal{O}_{M'}$ of sheaves of \mathbb{C} -algebras which commutes with the surjections to \mathcal{O}_M .*
- (ii) *The manifold-germ M' is unique up to unique isomorphism in the following sense. If we have two complex manifolds M'_1, M'_2 , two closed embeddings $\iota_1: M \rightarrow M'_1, \iota_2: M \rightarrow M'_2$, and an isomorphism of sheaves of \mathbb{C} -algebras $\phi: \iota_1^{-1}\mathcal{O}_{M'_1} \rightarrow \iota_2^{-1}\mathcal{O}_{M'_2}$ commuting with the surjections to \mathcal{O}_M , then ϕ is induced by a biholomorphic map $\varphi: N_2 \rightarrow N_1$ between open neighbourhoods N_i of M in M'_i , $i \in \{1, 2\}$, such that φ is the identity map on M . Such a map φ is unique as a germ of maps on a neighbourhood of M .*

We also discuss the extension of sheaves, proving:

Theorem 2. *Suppose that $\iota: M \hookrightarrow M'$ is a closed embedding of complex manifolds. Let \mathcal{A} be the sheaf of \mathbb{C} -algebras $\mathcal{A} = \iota^{-1}\mathcal{O}_{M'}$ over M and let \mathcal{B} be a coherent \mathcal{A} -module. Then:*

- (i) *There exist an open neighbourhood N of M in M' and a coherent \mathcal{O}_N -module \mathcal{B}' on N such that $\iota^{-1}\mathcal{B}' \cong \mathcal{B}$ as sheaves of \mathcal{A} -modules.*
- (ii) *The sheaf-germ \mathcal{B}' is unique up to unique isomorphism in the following sense. If we have two coherent \mathcal{O}_N -modules \mathcal{B}'_1 and \mathcal{B}'_2 on a neighbourhood N of M in M' and isomorphisms $\iota^{-1}\mathcal{B}'_1 \cong \mathcal{B} \cong \iota^{-1}\mathcal{B}'_2$ of \mathcal{A} -modules, with $\phi: \iota^{-1}\mathcal{B}'_1 \rightarrow \iota^{-1}\mathcal{B}'_2$ denoting the composite isomorphism, then ϕ is induced by an isomorphism $\Phi: \mathcal{B}'_1|_P \rightarrow \mathcal{B}'_2|_P$ of \mathcal{O}_P -modules on an open*

Key words and phrases. complex manifold, real analytic manifold, neighbourhood, germ, gluing, Hausdorff, complexification, unfolding, Frobenius manifold, TEP structure.

neighbourhood P of M in N . Such a morphism Φ is unique as a germ of homomorphisms over a neighbourhood of M .

Remark 3. By Oka's coherence theorem, $\mathcal{O}_{M'}$ is coherent and hence $\mathcal{A} = \iota^{-1}\mathcal{O}_{M'}$ is also coherent as a sheaf of algebras. Therefore being a coherent \mathcal{A} -module is equivalent to being *locally finitely presented* as an \mathcal{A} -module, i.e. for each $x \in M$, there exists an open neighbourhood U of x in M and an exact sequence of \mathcal{A} -modules:

$$(2) \quad \mathcal{A}_U^{\oplus k} \longrightarrow \mathcal{A}_U^{\oplus l} \longrightarrow \mathcal{B}|_U \longrightarrow 0$$

for some $k, l \in \mathbb{N}$. See e.g. [9, Appendix].

Remark 4. If in addition \mathcal{B} is locally free as an \mathcal{A} -module in Theorem 2, then \mathcal{B}' becomes locally free as an \mathcal{O}_N -module in a neighbourhood N of M , because the stalk \mathcal{B}'_x at each $x \in M$ is a free $\mathcal{O}_{M',x}$ -module.

Remark 5. We have stated Theorems 1 and 2 in the category of holomorphic manifolds, but the same statements hold true, with the same proofs, in the real analytic category.

We can reformulate our results as an *equivalence of categories*. Namely, for Theorem 1, the category of sheaves \mathcal{A} of \mathbb{C} -algebras on M equipped with surjections $\pi: \mathcal{A} \rightarrow \mathcal{O}_M$ satisfying the local condition (1) is equivalent to the category of germs of neighbourhoods $\iota: M \hookrightarrow M'$ of M . For Theorem 2, the category of coherent \mathcal{A} -modules is equivalent to the category of germs of coherent sheaves on a neighbourhood of M in M' . It is not difficult to modify the discussion below to prove these categorical equivalences.

In the real analytic category (Remark 5), Theorem 1 may be viewed as a generalization of the existence theorem for the complexification of a real-analytic manifold, see e.g. [12]. In the C^∞ category, Lemma 8 below is not valid (see e.g. [11]) and our results do not hold. However, most of the arguments for Theorem 1 work if we can take representatives of neighbourhood germs and C^∞ gluing maps between them which satisfy cocycle conditions as germs; similar arguments appear in the context of Kuranishi structures [5, 8]. Our original motivation was to globalize the unfolding of Frobenius-type structures (or meromorphic connections, or TEP structures) which has been studied by Hertling–Manin [7] and Reichelt [10] on the level of germs. We give a global version of Hertling–Manin's unfolding theorem for TEP structures in section 4 below.

Notation 6. We require that manifolds be paracompact.

Notation 7. We write $A \Subset B$ if and only if A is a relatively compact subset of B .

Acknowledgments. T.C. thanks Eugenia Cheng for useful correspondence. H.I. thanks Ono Kaoru and Ken-ichi Yoshikawa for useful discussions.

2. THE PROOF OF THEOREM 1

Our assumptions on \mathcal{A} imply that we can find a locally-finite open covering $\{W_i : i \in I\}$ of M with index set I such that $\mathcal{A}|_{W_i} \cong \iota^{-1}\mathcal{O}_{W_i \times Z_i}$, where Z_i is a copy of \mathbb{C}^n and $\iota: W_i \rightarrow W_i \times Z_i$ is the embedding $x \mapsto (x, 0)$. Without loss of generality we can assume both that W_i is a co-ordinate neighbourhood on M (i.e. is identified with an open subset of \mathbb{C}^m) and that W_i is relatively compact in M . We take locally finite coverings $\{U_i : i \in I\}$, $\{V_i : i \in I\}$ with the same index set I such that $U_i \Subset V_i \Subset W_i$. We write $W_{ij} := W_i \cap W_j$, $V_{ij} := V_i \cap V_j$, $V_{ijk} := V_i \cap V_j \cap V_k$. The basic fact we use for the gluing is the following Lemma.

Lemma 8. *Let $U \subset \mathbb{C}^m$ be an open set and let $\iota: U \rightarrow U \times \mathbb{C}^n$ be the embedding $x \mapsto (x, 0)$. Let $\phi: \iota^{-1}\mathcal{O}_{U \times \mathbb{C}^n} \rightarrow \iota^{-1}\mathcal{O}_{U \times \mathbb{C}^n}$ be a homomorphism of sheaves of \mathbb{C} -algebras which commutes with the natural surjections to \mathcal{O}_U . Then:*

- (a) *there exists an open neighbourhood U' of $U \times \{0\}$ in $U \times \mathbb{C}^n$ and a holomorphic map $\varphi: U' \rightarrow U \times \mathbb{C}^n$ which is the identity on $U \times \{0\}$ such that ϕ coincides with the pull-back by φ ;*
- (b) *if ϕ is an isomorphism then the map φ is a biholomorphic isomorphism onto its image.*

Proof of Lemma 8. Statement (a) implies statement (b), by the inverse function theorem, so we prove (a). Consider first the case where $m = 0$ and U is a point. Then ϕ is a \mathbb{C} -algebra endomorphism of the ring $\iota^{-1}\mathcal{O}_{\mathbb{C}^n} = \mathbb{C}\{z_1, \dots, z_n\}$ of convergent power series which preserves the maximal ideal $\mathfrak{m} = (z_1, \dots, z_n)$. Because such a ϕ is continuous with respect to the \mathfrak{m} -adic topology, ϕ is determined by the images of the generators z_1, \dots, z_n . The images determine a holomorphic map $\varphi: (z_1, \dots, z_n) \mapsto (\phi(z_1), \dots, \phi(z_n))$ which is defined on a neighbourhood U' of 0 in \mathbb{C}^n , and ϕ coincides with the pull-back by φ .

Consider now the general case. Let t_1, \dots, t_m denote the standard co-ordinates on $U \subset \mathbb{C}^m$ and let z_1, \dots, z_n denote the standard co-ordinates on \mathbb{C}^n . Then the images of t_1, \dots, t_m and z_1, \dots, z_n under ϕ give global sections of $\iota^{-1}\mathcal{O}_{U \times \mathbb{C}^m}$, and thus they define a holomorphic map

$$\varphi: (t_1, \dots, t_m, z_1, \dots, z_n) \mapsto (\phi(t_1), \dots, \phi(t_m), \phi(z_1), \dots, \phi(z_n))$$

on a neighbourhood U' of $U \times \{0\}$ in $U \times \mathbb{C}^n$. Since ϕ commutes with the surjections to \mathcal{O}_U , φ restricts to the identity map on $U \times \{0\}$. The pull-back $\varphi^*: \iota^{-1}\mathcal{O}_{U \times \mathbb{C}^n} \rightarrow \iota^{-1}\mathcal{O}_{U \times \mathbb{C}^m}$ defines a homomorphism of sheaves of \mathbb{C} -algebras commuting with the surjections to \mathcal{O}_U . The $m = 0$, $U = \text{point}$ case implies that $(\varphi^*)_t = \phi_t$ on the stalk at every point $t \in U$. Thus $\varphi^* = \phi$. \square

The composite isomorphism $\iota^{-1}\mathcal{O}_{W_{ij} \times Z_i} \cong \mathcal{A}|_{W_{ij}} \cong \iota^{-1}\mathcal{O}_{W_{ij} \times Z_j}$ induces a biholomorphic isomorphism $\varphi_{ij}: N_{ij} \rightarrow N_{ji}$ for each $i, j \in I$, where N_{ij} is an open neighbourhood of $W_{ij} \times \{0\}$ in $W_{ij} \times Z_i$ and φ_{ij} is the identity on $W_{ij} \times \{0\}$. Note that N_{ij} and N_{ji} are subsets of different spaces. Without loss of generality we may assume that $N_{ii} = W_i \times Z_i$ and that φ_{ii} is the identity map. Define:

$$\begin{aligned} O_i &:= V_i \times Z_i \\ O_{ij} &:= (V_{ij} \times Z_i) \cap N_{ij} \cap \varphi_{ij}^{-1}(V_{ij} \times Z_j) \end{aligned}$$

Then O_{ij} is an open subset of O_i which contains $V_{ij} \times \{0\}$. By restricting φ_{ij} to O_{ij} , we obtain a biholomorphic isomorphism $\varphi_{ij}: O_{ij} \rightarrow O_{ji}$ such that $\varphi_{ij}|_{V_{ij} \times \{0\}}$ is the identity map.

Lemma 9. *There exist open subsets Q_i of O_i , $i \in I$, such that for each $i, j \in I$ we have:*

- (a) $Q_i \subseteq O_i$;
- (b) $U_i \times \{0\} \subset Q_i \subset U_i \times Z_i$;
- (c) $Q_{ij} \subseteq O_{ij}$, where $Q_{ij} := Q_i \cap O_{ij} \cap \varphi_{ij}^{-1}(Q_j)$.

Proof of Lemma 9. Denote by \overline{U}_i the closure of U_i in V_i ; by assumption \overline{U}_i is compact. We have that $\overline{U}_i \cap \overline{U}_j$ is contained in V_{ij} , and hence that $(\overline{U}_i \cap \overline{U}_j) \times \{0\} \subset O_{ij}$. Fix $i, j \in I$, and fix a relatively compact open subset P of O_{ij} such that P contains $(\overline{U}_i \cap \overline{U}_j) \times \{0\}$; such a subset exists because the set $\overline{U}_i \cap \overline{U}_j$ is compact. Set:

$$Q_i(n) := U_i \times \{x \in Z_i : |x| < \frac{1}{n}\}$$

noting that $Q_i(n)$ satisfies conditions (a) and (b) of the Lemma.

We claim that there exists n such that

$$(3) \quad Q_i(n) \cap O_{ij} \cap \varphi_{ij}^{-1}(Q_j(n)) \subset P$$

Suppose, on the contrary, that for each n there exists an element $x_n \in Q_i(n) \cap O_{ij} \cap \varphi_{ij}^{-1}(Q_j(n))$ such that $x_n \notin P$. After passing to a subsequence, we have that (x_n) converges to a limit $x \in \overline{U}_i \times \{0\}$. Thus (x_n) converges in O_i . On the other hand each x_n lies in the closed subset $O_i \setminus P$ of O_i , and so $x \in O_i \setminus P$. Thus x lies in $(\overline{U}_i \setminus \overline{U}_j) \times \{0\}$. Now x_n lies in O_{ij} for each n , hence x_n lies in $V_{ij} \times Z_i$

and the limit x lies in the closure of V_{ij} in M . However the closure of V_{ij} is contained in W_{ij} . Recall that φ_{ij} is defined and continuous on the open neighbourhood N_{ij} of $W_{ij} \times \{0\}$ in W'_i . Thus:

$$\varphi_{ij}(x) = \lim_{n \rightarrow \infty} \varphi_{ij}(x_n)$$

The right hand side here converges to an element in $\overline{U}_j \times \{0\}$, since $\varphi_{ij}(x_n) \in Q_j(n)$. This is a contradiction: we have shown that $x \in \overline{U}_i \setminus \overline{U}_j$, and $\varphi_{ij}|_{W_{ij} \times \{0\}}$ is the identity map. Thus, for each i and $j \in I$, there exists an integer $n = n(i, j)$ such that (3) holds.

Since V_i is relatively compact in M and since the covering $\{V_i : i \in I\}$ is locally finite, only finitely many V_j have non-empty intersection with a fixed V_i . Thus we can define:

$$\begin{aligned} n(i) &:= \max\{n(i, j) : j \in I \text{ such that } V_i \cap V_j \neq \emptyset\} \\ Q_i &:= Q_i(n(i)) \end{aligned}$$

to obtain open sets $\{Q_i : i \in I\}$ with the properties claimed. \square

Lemma 10. *Let $\{Q_i : i \in I\}$ be such that Q_i is an open subset of O_i and that properties (a–c) in Lemma 9 hold. Then the image of the map $Q_{ij} \rightarrow Q_i \times Q_j$ given by $(\text{inclusion}, \varphi_{ij})$ is closed in $Q_i \times Q_j$.*

Proof of Lemma 10. Let (x_n) be a sequence in Q_{ij} such that (x_n) converges in Q_i and $(\varphi_{ij}(x_n))$ converges in Q_j . Let x denote the limit of (x_n) in Q_i . Since Q_{ij} is relatively compact in O_{ij} , the limit x lies in O_{ij} . But $(\varphi_{ij}(x_n))$ converges in Q_j and thus $\varphi_{ij}(x) \in Q_j$, or in other words $x \in \varphi_{ij}^{-1}(Q_j)$. Thus $x \in Q_{ij}$. \square

Lemma 11. *There exist open subsets Q_i of O_i , $i \in I$, such that properties (a–c) in Lemma 9 hold and further, for each $i, j, k \in I$, we have:*

- (d) $Q_{ij} \cap Q_{ik} \subset \varphi_{ij}^{-1}(O_{jk})$;
- (e) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $Q_{ij} \cap Q_{ik}$.

Note that condition (d) guarantees that the composition in (e) is well-defined.

Proof of Lemma 11. Let $\{Q_i : i \in I\}$ be such that, for each $i \in I$, Q_i is an open subset of O_i and that properties (a–c) hold. Such subsets exist by Lemma 9. Define:

$$Q_{ijk} := Q_{ij} \cap Q_{ik}$$

Let \overline{Q}_{ij} denote the closure of Q_{ij} in O_{ij} . This is compact, and hence \overline{Q}_{ij} is at the same time the closure of Q_{ij} in O_i . Let \overline{Q}_{ijk} denote the closure of Q_{ijk} in O_i . This is contained in the compact set $\overline{Q}_{ij} \cap \overline{Q}_{ik}$, hence in particular is contained in $O_{ij} \cap O_{ik}$.

We claim that there exists an open neighbourhood N_{ijk} of $\overline{Q}_{ijk} \cap (U_i \times \{0\})$ in $O_{ij} \cap O_{ik}$ such that:

- $N_{ijk} \subset \varphi_{ij}^{-1}(O_{jk})$; and
- $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ holds on N_{ijk} .

It suffices to show that each x in $\overline{Q}_{ijk} \cap (U_i \times \{0\})$ has an open neighbourhood N_x in $O_{ij} \cap O_{ik}$ such that $N_x \subset \varphi_{ij}^{-1}(O_{jk})$ and that $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ holds on N_x . Let $x \in \overline{Q}_{ijk} \cap (U_i \times \{0\})$. Then x lies in $O_{ij} \cap O_{ik} \cap (U_i \times \{0\}) = (V_{ijk} \cap U_i) \times \{0\}$ and hence lies in $O_{ij} \cap O_{ik} \cap \varphi_{ij}^{-1}(O_{jk})$. But $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ holds as germs at x , and so choosing N_x to be a sufficiently small open neighbourhood of x in $O_{ij} \cap O_{ik} \cap \varphi_{ij}^{-1}(O_{jk})$ proves the claim.

Recall that properties (a–c) in Lemma 9 hold for $\{Q_i : i \in I\}$ and note that, with the exception of the assertion that $U_i \times \{0\} \subset Q_i$, these properties are preserved under shrinking the sets Q_i . For a fixed $i \in I$, there are only finitely many pairs $(j, k) \in I \times I$ such that the triple intersection V_{ijk} is nonempty. Thus, as $Q_{ijk} \subset V_{ijk} \times Z_i$, there are only finitely many pairs $(j, k) \in I \times I$ such that

Q_{ijk} is nonempty. Let $(j_1, k_1), \dots, (j_f, k_f)$ be all such pairs. We shrink Q_i inductively as follows: set $Q_i^{(0)} := Q_i$, set

$$Q_i^{(a)} := (Q_i^{(a-1)} \setminus \overline{Q_{ij_1 k_1}}) \cup (N_{ij_1 k_1} \cap Q_i^{(a-1)}) \subset Q_i^{(a-1)}$$

for $a = 1, \dots, f$, and set $Q_i^{\text{new}} := Q_i^{(f)}$. Then Q_i^{new} is an open subset of the original set Q_i , and it contains $U_i \times \{0\}$. Thus properties (a–c) in Lemma 9 hold for the new sets $\{Q_i^{\text{new}} : i \in I\}$. Furthermore, for each $i, j, k \in I$, $Q_{ij}^{\text{new}} := Q_i^{\text{new}} \cap Q_{ij} \cap \varphi_{ij}^{-1}(Q_j^{\text{new}})$ is contained in Q_{ij} and $Q_{ijk}^{\text{new}} := Q_{ij}^{\text{new}} \cap Q_{ik}^{\text{new}}$ is contained in Q_{ijk} . Also Q_{ijk}^{new} is contained in N_{ijk} . Therefore properties (d) and (e) hold for $\{Q_i^{\text{new}} : i \in I\}$, and the Lemma is proved. \square

Remark. Let $Q_i \subset O_i$, $i \in I$, be open subsets such that properties (a–e) in Lemma 11 hold. In particular, then, $Q_{ij} \cap Q_{ik} \subset \varphi_{ij}^{-1}(O_{jk})$. But slightly more is true: in fact $Q_{ij} \cap Q_{ik} \subset \varphi_{ij}^{-1}(Q_{jk})$. For if $x \in Q_{ij} \cap Q_{ik}$ then $\varphi_{ij}(x) \in Q_{ji} \subset Q_j$ and $\varphi_{ik}(x) \in Q_{ki} \subset Q_k$. Also $\varphi_{jk} \circ \varphi_{ij}(x) = \varphi_{ik}(x)$, which lies in Q_k . Thus $\varphi_{ij}(x)$ lies in $Q_j \cap O_{jk} \cap \varphi_{jk}^{-1}(Q_k) =: Q_{jk}$.

We now complete the proof of Theorem 1. Choose open subsets $Q_i \subset O_i$, $i \in I$, such that properties (a–e) in Lemma 11 hold. (This is possible by Lemma 11.) Set M' equal to the quotient space:

$$\left(\coprod_{i \in I} Q_i \right) / \sim$$

by the equivalence relation \sim generated by $x \sim \varphi_{ij}(x)$ where $x \in Q_{ij}$. We claim that M' is a complex manifold.

Let $X = \coprod_{i \in I} Q_i$ and consider the binary relation R on $X \times X$ given by:

$$R := \coprod_{i, j \in I} Q_{ij} \subset \coprod_{i, j \in I} Q_i \times Q_j = X \times X$$

where the map $Q_{ij} \rightarrow Q_i \times Q_j$ is given by (inclusion, φ_{ij}). Then M' is the quotient space of X by the equivalence relation generated by R . To show that M' is a complex manifold it suffices to prove that M' is Hausdorff; hence it suffices to prove that R is closed and that R is an equivalence relation. We have shown that the image of the map $Q_{ij} \rightarrow Q_i \times Q_j$ is closed (Lemma 10), so R is closed. It remains to show that R is an equivalence relation. Reflexivity ($x \sim x$) is obvious, since $Q_{ii} = Q_i$ and φ_{ii} is the identity map. Symmetricity ($x \sim y \implies y \sim x$) is also obvious, since φ_{ij} and φ_{ji} are inverse to each other. For transitivity ($x \sim y \wedge y \sim z \implies x \sim z$) assume that $x \in Q_j$, $y \in Q_i$, $z \in Q_k$, $x \sim y$, and $y \sim z$. Then $y \in Q_{ij}$ (since $y \sim x$) and $y \in Q_{ik}$ (since $y \sim z$), thus $y \in Q_{ij} \cap Q_{ik}$. The Remark after Lemma 11 implies that $y \in \varphi_{ij}^{-1}(Q_{jk})$, and Lemma 11 implies that $\varphi_{jk} \circ \varphi_{ij}(y) = \varphi_{ik}(y)$. But $x = \varphi_{ij}(y)$ and $z = \varphi_{ik}(y)$, so $z = \varphi_{jk}(x)$. Thus $x \sim z$. It follows that R is an equivalence relation, and that M' is a complex manifold. It is clear that M is a closed submanifold of M' . This completes the proof of part (i) of Theorem 1.

Let us prove part (ii) of Theorem 1. Suppose we have two closed embeddings $\iota_1: M \rightarrow M'_1$, $\iota_2: M \rightarrow M'_2$ and an isomorphism $\phi: \iota_1^{-1}\mathcal{O}_{M'_1} \cong \iota_2^{-1}\mathcal{O}_{M'_2}$ of sheaves of \mathbb{C} -algebras commuting with natural surjections to \mathcal{O}_M . By Lemma 8, the isomorphism $\phi: \iota_1^{-1}\mathcal{O}_{M'_1} \rightarrow \iota_2^{-1}\mathcal{O}_{M'_2}$ is locally induced by a biholomorphic map which is the identity on M . Therefore we have a locally finite open covering $\{S_i : i \in I\}$ of M , open neighbourhoods T_i of S_i in M'_2 , and holomorphic maps $\varphi: T_i \rightarrow M'_1$ such that φ_i is the identity map on $T_i \cap M$ and $\phi|_{S_i} = \varphi_i^*$. Without loss of generality we may assume that S_i is relatively compact in M . Choose an open covering $\{R_i : i \in I\}$ of M such that $R_i \Subset S_i$, and choose an open tubular neighbourhood of M in M'_2 . The tubular neighbourhood here is identified with a neighbourhood of the zero section of the normal bundle of M in M'_2 ; we choose a (fibrewise) Riemannian metric on it. We write $U(\epsilon) \subset M'_2$ for the open tube of length $\epsilon > 0$ over an open subset $U \subset M$. The maps φ_i and φ_j on the overlap $T_{ij} := T_i \cap T_j$ coincide on an open neighbourhood

$T_{ij}^\circ \subset T_{ij}$ of $S_{ij} := S_i \cap S_j$. Since, for fixed $i \in I$, there are only finitely many $j \in I$ such that S_{ij} is nonempty, there exists $\epsilon_i > 0$ such that:

- $R_i(\epsilon_i) \subset T_i$
- $R_{ij}(\epsilon_i) \subset T_{ij}^\circ$ for all $j \in I$, where $R_{ij} := R_i \cap R_j \in S_{ij}$.

Then the maps $\{\varphi_i|_{R_i(\epsilon_i)} : i \in I\}$ coincide over each overlap $R_i(\epsilon_i) \cap R_j(\epsilon_j) \subset T_{ij}^\circ$, $i, j \in I$, and thus determine a global holomorphic map φ on $N = \bigcup_{i \in I} R_i(\epsilon_i) \subset M'_2$. The uniqueness of φ as a germ is obvious. This completes the proof of Theorem 1.

3. THE PROOF OF THEOREM 2

Lemma 12. *Let $\iota : M \rightarrow M'$ be a closed embedding of complex manifolds, let \mathcal{A} be the sheaf of \mathbb{C} -algebras $\mathcal{A} = \iota^{-1}\mathcal{O}_{M'}$ over M , and let \mathcal{B} be a locally finitely presented $\mathcal{A} = \iota^{-1}\mathcal{O}_{M'}$ -module. There exist:*

- an open covering $\{V_i : i \in I\}$ of M such that V_i is relatively compact in M ;
- for each $i \in I$, an open subset W'_i of M' such that $V_i \subset W'_i$;
- for each $i, j \in I$, an open subset A_{ij} of M' such that $V_{ij} \subset A_{ij} \subset W'_{ij}$, where $V_{ij} := V_i \cap V_j$ and $W'_{ij} := W'_i \cap W'_j$;
- for each $i, j, k \in I$, an open subset B_{ijk} of M' such that $V_{ijk} \subset B_{ijk} \subset A_{ijk}$, where $V_{ijk} := V_i \cap V_j \cap V_k$ and $A_{ijk} := A_{ij} \cap A_{jk} \cap A_{ik}$;
- for each $i \in I$, a coherent $\mathcal{O}_{W'_i}$ -module \mathcal{B}'_i on W'_i ;
- for each $i \in I$, an isomorphism $\theta_i : \iota^{-1}\mathcal{B}'_i \cong \mathcal{B}|_{V_i}$ of \mathcal{A}_{V_i} -modules;
- for each $i, j \in I$, an isomorphism $\phi_{ij} : \mathcal{B}'_i|_{A_{ij}} \cong \mathcal{B}'_j|_{A_{ij}}$ of $\mathcal{O}_{A_{ij}}$ -modules;

such that A_{ij}, B_{ijk} are symmetric in their indices and that the diagrams:

$$(4) \quad \begin{array}{ccccc} \iota^{-1}\mathcal{B}'_i|_{A_{ij}} & \xrightarrow{\iota^{-1}\phi_{ij}} & \iota^{-1}\mathcal{B}'_j|_{A_{ij}} & & \mathcal{B}'_i|_{B_{ijk}} \xrightarrow{\phi_{ik}} \mathcal{B}'_k|_{B_{ijk}} \\ & \searrow \theta_i & \swarrow \theta_j & & \searrow \phi_{ij} \quad \nearrow \phi_{jk} \\ & & \mathcal{B}|_{V_{ij}} & & \mathcal{B}'_j|_{B_{ijk}} \end{array}$$

commute for each $i, j, k \in I$.

Proof of Lemma 12. We take open coverings $\{V_i : i \in I\}, \{W_i : i \in I\}$ of M by Stein open subsets V_i, W_i such that $V_i \Subset W_i \Subset M$ and that $\mathcal{B}|_{W_i}$ has a finite presentation

$$\mathcal{A}_{W_i}^{\oplus k_i} \xrightarrow{\xi_i} \mathcal{A}_{W_i}^{\oplus l_i} \xrightarrow{\eta_i} \mathcal{B}|_{W_i} \longrightarrow 0$$

as in (2). For example, we can take V_i, W_i to be small open balls centered at the same point of a co-ordinate chart. The \mathcal{A} -module homomorphism $\xi_i : \mathcal{A}_{W_i}^{\oplus k_i} \rightarrow \mathcal{A}_{W_i}^{\oplus l_i}$ extends to an \mathcal{O} -module homomorphism $\xi'_i : \mathcal{O}_{W'_i}^{\oplus k_i} \rightarrow \mathcal{O}_{W'_i}^{\oplus l_i}$ on a neighbourhood W'_i of W_i in M' and defines a coherent $\mathcal{O}_{W'_i}$ -module \mathcal{B}'_i by the exact sequence:

$$\mathcal{O}_{W'_i}^{\oplus k_i} \xrightarrow{\xi'_i} \mathcal{O}_{W'_i}^{\oplus l_i} \xrightarrow{\eta'_i} \mathcal{B}'_i \longrightarrow 0$$

By construction there is an \mathcal{A} -module isomorphism $\theta_i : \iota^{-1}\mathcal{B}'_i \rightarrow \mathcal{B}|_{V_i}$. For each pair (i, j) such that $V_{ij} := V_i \cap V_j$ is nonempty, we construct a homomorphism ϕ_{ij} from \mathcal{B}'_i to \mathcal{B}'_j . Let e_1, \dots, e_{l_i} denote the standard basis of $\mathcal{O}_{W'_i}^{\oplus l_i}$. For each $1 \leq a \leq l_i$, the image $\eta_i(e_a)$ is a section of \mathcal{B}'_i and induces a section s_a of $\iota^{-1}\mathcal{B}'_i \cong \mathcal{B}|_{V_i}$. Via the isomorphism $\iota^{-1}\mathcal{B}'_j|_{V_{ij}} \cong \mathcal{B}|_{V_{ij}}$, the restriction $s_a|_{V_{ij}}$ can be lifted to a section t_a of \mathcal{B}'_j over an open neighbourhood C_{ij} of W_{ij} in W'_{ij} . Because the intersection V_{ij} of Stein open sets V_i, V_j is Stein and because $V_{ij} \Subset W_{ij}$, we can find a Stein

open neighbourhood A_{ij} of V_{ij} in C_{ij} . Because A_{ij} is Stein, we can find a lift $u_a \in \Gamma(A_{ij}, \mathcal{O}^{\oplus l_j})$ of $t_a|_{A_{ij}}$ such that $\eta_j(u_a) = t_a|_{A_{ij}}$. The sections u_1, \dots, u_{l_i} define a homomorphism $\psi_{ij}: \mathcal{O}_{A_{ij}}^{\oplus l_i} \rightarrow \mathcal{O}_{A_{ij}}^{\oplus l_j}$ sending e_a to u_a .

$$\begin{array}{ccccccc} \mathcal{O}_{A_{ij}}^{\oplus k_i} & \xrightarrow{\xi_i} & \mathcal{O}_{A_{ij}}^{\oplus l_i} & \xrightarrow{\eta_i} & \mathcal{B}'_i|_{A_{ij}} & \longrightarrow & 0 \\ & & \downarrow \psi_{ij} & & & & \\ \mathcal{O}_{A_{ij}}^{\oplus k_j} & \xrightarrow{\xi_j} & \mathcal{O}_{A_{ij}}^{\oplus l_j} & \xrightarrow{\eta_j} & \mathcal{B}'_j|_{A_{ij}} & \longrightarrow & 0 \end{array}$$

We claim that, after shrinking A_{ij} if necessary, ψ_{ij} induces a homomorphism $\phi_{ij}: \mathcal{B}'_i|_{A_{ij}} \rightarrow \mathcal{B}'_j|_{A_{ij}}$. It suffices to show that the composition $\eta_j \circ \psi_{ij} \circ \xi_i$ is zero in a neighbourhood of V_{ij} . By construction, $\eta_i(v)$ and $\eta_j \circ \psi_{ij}(v)$ define the same section of \mathcal{B} over V_{ij} for every $v \in \Gamma(A_{ij}, \mathcal{O}_{A_{ij}}^{\oplus l_i})$. Hence for $w \in \Gamma(A_{ij}, \mathcal{O}_{A_{ij}}^{\oplus k_i})$, $\eta_j \circ \psi_{ij} \circ \xi_i(w)$ and $\eta_i \circ \xi_i(w) = 0$ define the same section of \mathcal{B} over V_{ij} . This means that $\eta_j \circ \psi_{ij} \circ \xi_i(w)$ vanishes in a neighbourhood of V_{ij} , and the claim follows. It is clear that the first diagram in (4) commutes. We can also assume that $A_{ij} = A_{ji}$ by replacing A_{ij} with $A_{ij} \cap A_{ji}$ and restricting ϕ_{ij} to it if necessary.

Finally we find an open subset B_{ijk} of $A_{ijk} = A_{ij} \cap A_{jk} \cap A_{ki}$ containing $V_{ijk} = V_i \cap V_j \cap V_k$ on which the second diagram in (4) commutes (i.e. the cocycle condition holds). But this is straightforward, because $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ holds at the stalk of each $x \in V_{ijk}$. \square

Let $\iota: M \hookrightarrow M'$, \mathcal{A} , and \mathcal{B} be as in the statement of Theorem 2. Take the data constructed in Lemma 12. By taking a refinement if necessary, we may assume that the open covering $\{V_i: i \in I\}$ is locally finite. Choose an open covering $\{U_i: i \in I\}$ of M such that $U_i \subseteq V_i$. Take an open tubular neighbourhood of M in M' and, as in the proof of Theorem 1, fix a fibrewise Riemannian metric on it. For a open subset U of M and $\epsilon > 0$, we denote by $U(\epsilon) \subset M'$ the open tube of length ϵ over U . For each $i \in I$, there are only finitely many $j \in I$ such that the intersection V_{ij} is non-empty. Therefore we can find $\epsilon_i > 0$ such that:

- $U_i(\epsilon_i) \subset W'_i$;
- $U_{ij}(\epsilon_i) \subset A_{ij}$ for all $j \in I$, where $U_{ij} := U_i \cap U_j$;
- $U_{ijk}(\epsilon_i) \subset B_{ijk}$ for all $j, k \in I$, where $U_{ijk} := U_i \cap U_j \cap U_k$.

Then the coherent sheaves $\mathcal{B}'_i|_{U_i(\epsilon_i)}$ glue together via the homomorphisms $\phi_{ij}|_{U_i(\epsilon_i) \cap U_j(\epsilon_j)}$ to give a global coherent \mathcal{O}_N -module \mathcal{B}' on $N = \bigcup_{i \in I} U_i(\epsilon_i)$. It is clear that there is an isomorphism $\iota^{-1}\mathcal{B}' \cong \mathcal{B}$ of \mathcal{A} -modules. This completes the proof of part (i) of Theorem 2.

Let us prove part (ii) of Theorem 2. Suppose that we have coherent \mathcal{O}_N -modules \mathcal{B}'_1 and \mathcal{B}'_2 and a isomorphisms $\iota^{-1}\mathcal{B}'_1 \cong \mathcal{B} \cong \iota^{-1}\mathcal{B}'_2$ of \mathcal{A} -modules. Let $\phi: \iota^{-1}\mathcal{B}'_1 \rightarrow \iota^{-1}\mathcal{B}'_2$ denote the composite isomorphism. We can find a locally finite open covering $\{S_i: i \in I\}$ of M together with a family $\{T_i: i \in I\}$ of open subsets of M' such that $S_i \subseteq T_i$ and $\mathcal{B}'_1|_{T_i}$ and $\mathcal{B}'_2|_{T_i}$ have finite presentations as \mathcal{O}_{T_i} -modules. Without loss of generality we may assume that S_i is Stein. The argument in the proof of Lemma 12 shows that we can find an open neighbourhood T_i° of S_i in T_i and a homomorphism $\Phi_i: \mathcal{B}'_1|_{T_i^\circ} \rightarrow \mathcal{B}'_2|_{T_i^\circ}$ such that $\iota^{-1}\Phi_i = \phi|_{S_i}$. For each $i, j \in I$, there exists an open subset P_{ij} of $T_i^\circ \cap T_j^\circ$ containing $S_{ij} := S_i \cap S_j$ such that $\Phi_i|_{P_{ij}} = \Phi_j|_{P_{ij}}$. Take an open covering $\{R_i: i \in I\}$ of M such that $R_i \subseteq S_i$. As before, we choose a tubular neighbourhood of M in M' and a (fibrewise) Riemannian metric on it, denoting by $U(\epsilon)$ the open ϵ -tube over the open subset $U \subset M$. Then for each $i \in I$, there exists $\epsilon_i > 0$ such that:

- $R_i(\epsilon_i) \subset T_i^\circ$;
- $R_{ij}(\epsilon_i) \subset P_{ij}$;

because there are only finitely many $j \in I$ such that S_{ij} is nonempty. Now the homomorphisms $\Phi_i|_{R_i(\epsilon_i)}$ glue together to define a global homomorphism $\Phi: \mathcal{B}'_1|_P \rightarrow \mathcal{B}'_2|_P$ over $P = \bigcup_{i \in I} R_i(\epsilon_i)$ such that $\phi = \iota^{-1}\Phi$. The uniqueness of Φ as a germ is obvious. This completes the proof of Theorem 2.

4. GLOBAL UNFOLDING OF TEP STRUCTURES

As an application of our results we now prove a global unfolding theorem for TEP structures, by globalizing the reconstruction theorem for germs of TEP structures due to Hertling–Manin [7]. This global unfolding theorem has applications in mirror symmetry and the study of quantum cohomology [1, §8.1.6]. TEP structures were introduced by Hertling [6]; they are closely related to Dubrovin’s notion of Frobenius manifold [3, 4, 7].

Definition 13 (TEP structure). Let M be a complex manifold. A *TEP structure* $(\mathcal{F}, \nabla, (\cdot, \cdot)_{\mathcal{F}})$ with base M consists of a locally free $\mathcal{O}_{M \times \mathbb{C}}$ -module \mathcal{F} of rank $N + 1$, and a meromorphic flat connection

$$\nabla: \mathcal{F} \rightarrow (\pi^* \Omega_M^1 \oplus \mathcal{O}_{M \times \mathbb{C}} z^{-1} dz) \otimes_{\mathcal{O}_{M \times \mathbb{C}}} \mathcal{F}(M \times \{0\})$$

so that for $f \in \mathcal{O}_{M \times \mathbb{C}}$, $s \in \mathcal{F}$, and tangent vector fields $v_1, v_2 \in \Theta_{M \times \mathbb{C}}$:

$$\nabla(fs) = df \otimes s + f \nabla s \quad [\nabla_{v_1}, \nabla_{v_2}] = \nabla_{[v_1, v_2]}$$

together with a non-degenerate pairing

$$(\cdot, \cdot)_{\mathcal{F}}: (-)^* \mathcal{F} \otimes_{\mathcal{O}_{M \times \mathbb{C}}} \mathcal{F} \rightarrow \mathcal{O}_{M \times \mathbb{C}}$$

which satisfies

$$\begin{aligned} ((-)^* s_1, s_2)_{\mathcal{F}} &= (-)^* ((-)^* s_2, s_1)_{\mathcal{F}} \\ d((-)^* s_1, s_2)_{\mathcal{F}} &= ((-)^* \nabla s_1, s_2)_{\mathcal{F}} + ((-)^* s_1, \nabla s_2)_{\mathcal{F}} \end{aligned}$$

for $s_1, s_2 \in \mathcal{F}$. Here $\mathcal{F}(M \times \{0\})$ denotes the sheaf of sections of \mathcal{F} with poles of order at most 1 along the divisor $M \times \{0\} \subset M \times \mathbb{C}$.

Definition 14 (Miniversality). Let M be a complex manifold. A *TEP structure* $(\mathcal{F}, \nabla, (\cdot, \cdot)_{\mathcal{F}})$ with base M is called *miniversal* if for each $y \in M$, the set

$$\{x \in \mathcal{F}|_{(y,0)} : \text{the map } T_y M \rightarrow \mathcal{F}|_{(y,0)}, v \mapsto (z \nabla_v x)|_{(y,0)} \text{ is an isomorphism}\}$$

is non-empty in the fiber $\mathcal{F}|_{(y,0)}$.

Theorem 15 (Global unfolding for TEP structures). *Let M be a complex manifold and $(\mathcal{F}, \nabla, (\cdot, \cdot)_{\mathcal{F}})$ a TEP structure with base M . Suppose that for each $y \in M$, there exists a section ζ of \mathcal{F} over a neighbourhood of $(y, 0) \in M \times \mathbb{C}$ such that:*

- (IC): *the map $T_y M \rightarrow \mathcal{F}|_{(y,0)}$ defined by $v \mapsto z \nabla_v \zeta|_{(y,0)}$ is injective;*
- (GC): *the fiber $\mathcal{F}|_{(y,0)}$ is generated by iterated derivatives*

$$(z^2 \nabla_{\partial_z})^l z \nabla_{v_1} \cdots z \nabla_{v_k} \zeta|_{(y,0)} \quad l \geq 0$$

with respect to local vector fields v_1, \dots, v_k on M near y and $z^2 \partial_z$.

Then there exist a complex manifold M' , a miniversal TEP structure $(\mathcal{F}', \nabla', (\cdot, \cdot)_{\mathcal{F}'})$ with base M' , and a closed embedding $\iota: M \rightarrow M'$ such that:

$$\iota^* (\mathcal{F}', \nabla', (\cdot, \cdot)_{\mathcal{F}'}) = (\mathcal{F}, \nabla, (\cdot, \cdot)_{\mathcal{F}})$$

Furthermore the manifold-germ M' and the TEP structure $(\mathcal{F}', \nabla', (\cdot, \cdot)_{\mathcal{F}'})$ are unique up to unique isomorphism in the sense of Theorem 1 and 2.

Proof. Combine Theorems 1 and 2 with the universal unfolding theorem for germs of TEP structures proved by Hertling–Manin [7, Theorem 2.5, Lemma 3.2]. \square

Analogous global unfolding theorems for TE structures [7], log-trTLEP structures [10], and so on can be proved in exactly the same way. Global unfoldings of log-trTLEP structures have interesting applications in Gromov–Witten theory [2].

REFERENCES

1. Tom Coates and Hiroshi Iritani. A Fock sheaf for Givental quantization. in preparation.
2. Tom Coates and Hiroshi Iritani. Modularity of Gromov–Witten potentials for local \mathbb{P}^2 . In preparation.
3. Boris Dubrovin. Geometry of 2D topological field theories. In *Integrable systems and quantum groups (Montecatini Terme, 1993)*, volume 1620 of *Lecture Notes in Math.*, pages 120–348. Springer, Berlin, 1996.
4. Boris Dubrovin. Geometry and analytic theory of Frobenius manifolds. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, number Extra Vol. II, pages 315–326, 1998.
5. Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Technical details on Kuranishi structure and virtual fundamental chain. [arXiv:1209.4410 \[math.SG\]](https://arxiv.org/abs/1209.4410), 2012.
6. Claus Hertling. tt^* geometry, Frobenius manifolds, their connections, and the construction for singularities. *J. Reine Angew. Math.*, 555:77–161, 2003.
7. Claus Hertling and Yuri Manin. Unfoldings of meromorphic connections and a construction of Frobenius manifolds. In *Frobenius manifolds*, Aspects Math., E36, pages 113–144. Vieweg, Wiesbaden, 2004.
8. Dominic Joyce. D-manifolds and d-orbifolds: a theory of derived differential geometry. Available at <http://people.maths.ox.ac.uk/joyce/dmbook.pdf>, 2012.
9. Masaki Kashiwara. *D-modules and microlocal calculus*, volume 217 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2003. Translated from the 2000 Japanese original by Mutsumi Saito, Iwanami Series in Modern Mathematics.
10. Thomas Reichelt. A construction of Frobenius manifolds with logarithmic poles and applications. *Comm. Math. Phys.*, 287(3):1145–1187, 2009.
11. Masahiro Shiota. Some results on formal power series and differentiable functions. *Publ. Res. Inst. Math. Sci.*, 12(1):49–53, 1967/77.
12. H. Whitney and F. Bruhat. Quelques propriétés fondamentales des ensembles analytiques-réels. *Comment. Math. Helv.*, 33:132–160, 1959.

E-mail address: `t.coates@imperial.ac.uk`

E-mail address: `iritani@math.kyoto-u.ac.jp`